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THE MATHEMATICS TEACHER

Volume XXIV Number 7

Edited by William David Reeve

High School Mathematics Teachers in the United States

By EARL W. ANDERSON *and* R. H. ELIASSEN

Bureau of Educational Research
Ohio State University
Columbus, Ohio

PROSPECTIVE TEACHERS and their advisers are often searching for information regarding the extent to which a specific subject is taught in high schools, the degree to which it is combined with other subjects, and what those other subjects are. It was the purpose of this study to bring together such reported findings relative to the teaching of mathematics.

A search of literature revealed analyses of high school teaching loads covering 25 areas, state-wide or larger in extent, including 98,464, or 46 per cent, of the teaching positions. One study covered positions in 14 large cities; another was a study of the South; 23 were state-wide studies covering 22 states. Most of the studies reported were made during the past four years but one went back to 1919.

One or more classes in mathematics were taught by 17,044, or 17 per cent, of the high school teachers. Table I gives the areas covered by each study, the year made, the number of teachers involved, and the number and per cent of teachers in each area teaching mathematics.

From Table I rather wide variations in the per cents of teachers who taught mathematics are seen. These run from 5 per cent in West Virginia—a surprisingly low proportion—to 38 per cent in North Dakota. The modal per cent was between 15 and 25. It would be expected that analyses of teaching loads in large city high schools or in states that were heavily urban would show smaller per cents teaching

TABLE I. THE NUMBER AND PER CENT OF MATHEMATICS TEACHERS FOUND IN 25 ANALYSES OF TEACHING POSITIONS

Area Studied	Year	Total No. of Teachers	No. of Mathematics Teachers	% of Mathematics Teachers
Indiana	1929–30	8,677	1,587	18
Missouri	1925–26	2,440	654	27
Illinois	1924–25	5,251	939	18
Idaho	1924–25	497	105	21
South Dakota	1924–25	1,255	385	31
Ohio	1928–29	9,100	2,366	26
Oklahoma	1925–26	2,903	605	21
Iowa	1924–25	1,478	228	15
Pennsylvania	1926–27	8,197	1,701	21
South	1926–27	11,472	1,720	15
New York	1927–28	11,395	1,445	13
Washington	1918–19	473	80	17
Michigan	1924–25	2,506	363	15
Minnesota	1921–22	1,134	251	22
Kansas	1924–25	2,449	632	26
North Dakota	1927–28	673	246	38
Wisconsin	1929–30	1,267	192	15
Arkansas	1929–30	423	87	21
Louisiana	1925–26	1,482	502	34
Maine	1925–26	912	200	22
Colorado	1926–27	532	111	21
California	1929–30	814	60	7
West Virginia	1929–30	482	26	5
Small H.S., West Va.	1929–30	377	121	32
13 Cities	1928	22,307	2,438	11

mathematics than would studies including mostly rural areas. This comes about because the city schools offer a more diversified course of study and because teachers in larger schools are more likely to be teaching mathematics exclusively than are those in the smaller ones. In the three-teacher high school often most teachers care for five different subjects, and one-third of the teaching staff teaches one or more classes in each. The large per cents of small schools make the per cent teaching mathematics in such predominantly rural states as North

TABLE II. THE EXTENT TO WHICH MATHEMATICS WAS TAUGHT ALONE AND IN COMBINATION WITH ONE OR MORE OTHER SUBJECTS, AS REPORTED BY 23* INVESTIGATORS

Area Studied	Year	Teaching Math. Alone	%	In Combination with Other Subjects									
				1	%	2	%	3	%	4	%	5	%
Indiana	29–30	586	37	674	42	244	15	68	4	15	1		
Missouri	25–26	239	37	275	42	98	15	37	5	5	1		
Illinois	24–25	374	40	423	45	118	13	22	2	2	−1		
Idaho	24–25	48	46	36	34	18	17	3	3				
South Dakota	24–25	56	15	178	46	136	35	15	4	2	1		
Ohio	28–29	574	24	500	21	568	24	457	19	218	9	39	2
Oklahoma	25–26	149	25	305	50	50	8						
Iowa	24–25	81	35	51	22	10	4						
Pennsylvania	26–27	739	43	727	43	209	12	26	2				
South	26–27	1,247	72	473	28								
New York	27–28	974	67	471	33								
Washington	18–19	36	45	7	9	37	46						
Michigan	24–25	245	67	93	26	24	7	0	0				
Minnesota	21–22	69	27	70	28	64	26	35	14	10	4	3	1
Kansas	24–25	116	18	135	21	188	30	147	23	45	7	1	−1
North Dakota	27–28	8	3	26	11	46	19	75	31	55	22	36	14
Wisconsin	29–30	61	32	96	50	35	18						
Arkansas	29–30	21	24	33	38	33	38						
Louisiana	25–26	129	26	223	45	130	26						
Maine	25–26	59	29	114	57	29	15						
Colorado	26–27	30	27	43	39	38	34						
California	29–30	8	13	36	60	10	17	5	8	1	1		
West Virginia	29–30	5	19	8	31	5	19	7	27	1	4		

* Two studies made no reports concerning combinations taught with mathematics.

Dakota and Louisiana higher than in the more thickly populated states such as Illinois, Pennsylvania, and New York.

A question commonly asked by those in teacher training and by prospective teachers is, to what extent are subjects taught alone and in combination with one, two, three, or more other subjects. Such data for mathematics are presented in Table II.

Table II shows a range in the per cents who taught mathematics alone from three in North Dakota to 67 in Michigan and New York. The Michigan study was made six years ago but it is hard to explain why so much larger percentage of the teachers taught mathematics alone in Michigan than in other similar states such as Ohio, Pennsyl-

TABLE III. A LIST OF THE MOST FREQUENT TEACHING COMBINATIONS WITH MATHEMATICS

Combinations	Per cent
Mathematics-Science	45.0
Mathematics-Social Studies	27.0
Mathematics-Physical Education	9.6
Mathematics-English	7.7
Mathematics-Latin	6.1
Mathematics-Commercial	2.3
Mathematics-Music	1.1
Mathematics-Agriculture	1.1

Read as follows: Of the subject combinations including mathematics, 45 per cent combined mathematics with science, 27 per cent with social studies, etc.

vania, and Wisconsin. There were surprisingly large per cents of teachers carrying three or more subjects other than mathematics.

The subjects most often combined with mathematics in high school teaching appear in the following order: science, social studies, physical education, Latin, English, agriculture, commercial studies, and music. By far the most common was science. Table III gives these data more in detail.

Most of the studies reported concerned all of the teachers in the state although a few of them covered beginning teachers only. It is likely, however, that a larger percentage of the beginning teachers will be found teaching other subjects with mathematics than is reported in these studies since the general tendency is for the more experienced teacher to get into the city schools in which single teaching subjects are common.

Summary

Twenty-five investigations, covering 46 per cent of the high school teaching loads in the United States, showed that 17 per cent of the high school teachers cared for one or more classes in mathematics. These per cents varied from 5 to 38 in different areas covered. Rural areas had larger per cents of the staff engaged in teaching mathematics than did more urban ones. The per cents who taught mathematics exclusively varied from 3 to 67. The more rural areas contained larger per cents teaching three or more subjects with mathematics. Science, social studies, physical education, and Latin occur most often with mathematics, in the order named.

National Council Yearbooks

Every teacher of mathematics who does not buy a copy of each of the National Council yearbooks for his personal use should bend every effort to see to it that a complete set of these books is placed in his own high school library or if that is not possible in the public library or some other available place. These yearbooks are very valuable and their value will increase as the years go by.

The First Yearbook is now out of print and the Council is being asked by large mathematics clubs to reprint this volume. This may be possible if enough teachers will promise to purchase a copy. However, the best way to be sure of obtaining these books is to buy them before the original edition is exhausted. The cost of reprinting may be exorbitant. A complete list of the yearbooks with their titles and price can be found in this issue on the page facing the front inside cover.

—THE EDITOR

An Experiment in Classification of Students in Mathematics

By S. Helen Taylor
University High School, Urbana, Illinois

The Problem

Beginning in the second semester, February, 1927, and continuing through four years, the mathematics department of the University of Illinois undertook a study of the effects of classification of students according to their mathematical ability. Several other departments have been interested in the same general problem, but the experiment in mathematics has been more extensive than has any other. It has involved a large number of students and a correspondingly large number of instructors. The experiment has also been extended over a much longer period of time than is usual.

It was not until the experiment was well started that first year that I became interested in the problem and accepted the responsibility of obtaining and working over the data contained in this paper. Unlike many pieces of experimental research, this study did not have at its inception the main purpose of satisfying one of the requirements for an advanced degree.

The particular experiment, though concerned with the general problem of the effects of classifying students according to ability in mathematics, has been made in relation to analytical geometry. By a choice of this course, it was possible to use the grades made by students in their university courses in algebra and trigonometry, thus insuring a more uniform basis of classification than by dependence on high school mathematics grades. A large group was available each year, for with the restriction of the experiment to the second semester when registration in analytics is much heavier, about four hundred students, registered in sixteen or eighteen sections, were included in each year's experiment. Complete statistical data have been collected for fourteen hundred students, and the preliminary tests have given data on more than fifteen hundred students. The number of instructors for the four years the experiment has been under way totals sixty-six or the same number as the total number of sections.

Counting out duplicates, for several instructors were in two years of the experiment, we have had fifty-five different instructors.

In February, 1927, there were 150 pupils in analytics grouped so that seventy-five were registered at nine o'clock and seventy-five at ten o'clock. These students at one hour were from the college of engineering and at the other from liberal arts and commerce. At each of these two hours three sections were made: high, middle, and low, on the basis of grades received in college algebra and trigonometry. The students whose grades in these courses were both A's, or both B's, or one A and one B, were put in the high section at the hour in question. Students whose grades were B and C, A and C, or both C's, were put in the middle section, while those with one C and one D, or with two D's were placed in the low section. There were 146 students in six sections at other hours where no grouping was attempted. These six control sections were the result of the ordinary process of random registration.

There were also eight sections, two at each of four different hours, divided into high and low sections. The high sections included those whose algebra-trigonometry average was B or above. The low sections included those whose average was C or D. These eight sections in 1927 were not given the preliminary tests, since they were not considered a part of the experiment, and for that reason quantitative results for purposes of comparison are not available. However, in the three years following, the sections having the two-way classification were given all the tests, and something will be said concerning this part of the plan in the later discussion of results.

The Procedure

The Iowa placement tests, one a mathematics training test, the other a mathematical aptitude test, were given early in one semester each year to all the pupils in the control and in the experimental sections. The department had these tests graded, and the grades were given out to the instructors, but the papers were kept on file. The scores from the Iowa tests were used to determine which students in the control and experimental sections could be considered as having equivalent mathematical training and equal mathematical aptitude. For example, in February last year there were, in the experimental sections grouped three at an hour, nine students whose scores in the two Iowa tests combined fell between the scores 90 and 95. There

were, in the experimental sections grouped two at an hour, twelve students with Iowa scores between 90 and 95. In the control sections there were ten students whose scores fell between 90 and 95. These thirty-one students were considered equal in training and aptitude and were compared at the end of the semester to determine whether the nine from the three-way classification and the twelve from the two-way classification did better or poorer work than did the ten in the control group as measured by the June tests in analytical geometry.

Each year, several times during the semester, students were interchanged between sections at the same hour in the groups where three sections met at an hour. This shifting was made on the basis of results of combined tests with committee grading, together with agreement on the part of the three instructors having sections at the same hour. This regular plan of shifting has been used only in those experimental sections involving three grades of ability and has never been extended to the two-way groups. We now feel that this shifting was advantageous on the whole and that perhaps it might have made for a more homogeneous grouping if it had been applied to these other experimental sections. This point will be considered in more detail later.

In the first year, shifts were made at the end of four weeks, eight weeks and twelve weeks. With the original personnel of the class and the three changes in personnel, the instructor's task of knowing anything about his student's work was very difficult. This was particularly true for the middle sections from which exchanges were made both upward and downward. The instructor in one of the middle sections had ten new students on the basis of this first change, although at the other hour only five new students came into the middle section. The average number of new students for each of the six classes at the first shift was five and one-half, five at the second shift, and only one at the third shift. The shifting in the engineering sections involved only one-half as many cases as in the sections of liberal arts and commerce students. A few students spent four weeks in each of the three sections, but the interchanges were usually not so radical or so frequent. However, it was decided to reduce the number of shifts, and therefore in the second and third years of the experiment changes were made at the end of six and twelve weeks. In the fourth and last year of the experiment only one regular shift was made and that at the end of six weeks.

At the end of the semester the final examination was a combined section examination given to all classes in analytics. This final examination consisted of two parts, the first of which we have designated as a new type test and the second as an old type test. The new type test was made in committee four years ago and then it was carefully revised on the basis of a critical study of each item on the four hundred papers from the first trial. This test was at first timed for an hour and twenty minutes, but in later years given in an hour. This test was following by the usual old type of examination which consisted of six questions on analytical geometry of the standard developmental type, five to be chosen from the six. The results of these June tests for students in the experimental and control sections who were deemed equal in ability by reason of comparable Iowa test scores, furnish us with data for our conclusions as to the effects of classification.

Closely related to the main problem, that of ascertaining the effects of classification on the basis of grades, we have recognized a number of other problems. A study of these other questions has thrown considerable light on the central problem. Among these are:

1. The determination of a suitable new type test in analytics.
2. A study of the correlation between the new type and old type tests, which are used together as an end test.
3. A comparison of the norms made in our classes in analytics in the Iowa placement tests with those recorded in the manual of directions which were set by classes just beginning algebra and trigonometry. This question may be stated in this form, "Have five hours of college mathematics given an appreciable amount of training or an added aptitude?"
4. A study of the difference in classification that would occur if the placement tests were used as a basis of the distribution into sections of analytics instead of the algebra and trigonometry grades.
5. To find by several years' experimenting with a change of instructors if the effects or results are independent of the teachers chosen.

The New Type Test in Analytical Geometry

There is available very little material which is useful in setting up a new type test in any branch of college mathematics. The Iowa tests concern high school mathematics only and hence a study of the details of these tests is not very suggestive. However, we wished to develop a test which could include many parts of the short answer

kind and which could be graded objectively. We wished to use it in conjunction with the more usual kind of examination, feeling that if the results in each case were in the same direction we would have convinced the two opposing groups in the testing movement that our results were valid.

A large number of items of the short answer type, the recognition type, and the true-false type were turned in by several members of the department who served as a committee to construct the test. These items were listed and then examined critically, as to form of statement and probable difficulty, by ten or twelve instructors. The test items as given the first year were thus organized solely on the basis of concensus of opinion of experts. The test has four parts, the first two each include twenty short problems or questions and the third and fourth parts each include twenty-five items, thus bringing the total up to ninety items. Twenty minutes per page is allowed, as the test now stands, making the time of the administration of the test an hour and twenty minutes.

On the first page each of the twenty statements describes a curve, and the students are asked to write the correct equation on the dotted line. These statements selected at random illustrate the kind of question used.

1. The straight line through the origin whose inclination is 135°. .....

2. The locus of points whose abscissas and ordinates are equal.

5. The circle with center at (2, 3) and radius 5. .....

9. The hyperbola with center at (1, 1) which passes through (3,-4). .....

11. The polar equation of a circle with center at pole and radius 5. .....

The second page consists of problems for which numerical answers are to be written in. The following questions illustrate this page.

Find the value of:

1. The distance from (1, 2) to (5,-5). .....

2. The y-coordinate of the midpoint of the line segment (1, 2), (5,-5). .....

5. The distance of the point (2, 1) from the line $3x + 4y + 5 = 0$. .....

13. The value of k if the straight line $2x - 3y + k = 0$ passes through the point (1, 1). .....

The third part contains twenty-five true-false statements to be identified. The fourth page is divided into two types of material. The first of these is a test to determine the student's ability to read new mathematical material and answer questions upon it. The second part of this last page lists fifteen equations of curves in a plane or in space and these are to be identified.

The four hundred papers from the first year's experiment were carefully analyzed and the errors tabulated. The number of times that each item of the ninety items was missed was thus recorded. In each page there were many items of no great difficulty for which the entire number of failures was only 12, 13, 18, 27, 56, or 61, while for the more difficult items the number of failures was as many as 228, 238, or 247. The test was revised and the items on each page arranged in the order of their relative difficulty. A few questions were reworded because of an obvious lack of precision. The time element for each part was shortened by five minutes as already indicated. The choice of reading material for the fourth part has been difficult and this material was twice changed in the course of the experiment. Whether this new type test serves its predetermined purpose of measuring the same ability or achievement in analytics as is measured by an old type test, with the added advantage of furnishing an objective scale for this measurement, can best be discussed in a consideration of our next question, that of the correlation of the new type and the old type tests.

Correlation of the June Tests

A study of the correlation between the two tests used in June as end tests has shown a correlation each year of approximately $.70 \pm .02$. The lowest correlation was in 1929 with a value of $.69 \pm .02$ and the highest was in 1930 with a value of $.71 \pm .02$. When one remembers that, for about fifty-five per cent of the standardized tests which are well thought of and widely used, the correlation between two forms of the same test lies between .70 and .90, then this correlation of .70 for two such different tests seems to indicate a satisfactory relation between them. We feel that the new type test has been constructed in a fairly satisfactory manner.

The Iowa Placement Tests

No doubt many of you are familiar with the Iowa placement tests, copyrighted in 1925, by the Extension Division of the University of

Iowa. Among other uses we find listed in the manual these two: to aid in selecting and admitting students, and to section classes for instructional purposes on the basis of mental ability. The norms stated in the manual are for groups where the tests were given at the end of high school work in mathematics, or rather at the very beginning of college work. We immediately wondered if such norms would hold for our students who were given the examination after having the additional five hours of training in mathematics. The experimental data lead us to believe that this five hours of mathematics has given a rather appreciable amount of training and an added aptitude, because our scores in general show a twenty-five per cent increase over the Iowa norms. We found each year that for the aptitude test, our lower quartile score was approximately the Iowa median, our median was their upper quartile and our upper quartile about twenty-five per cent above their upper quartile. In the training test our norms are approximately thirty per cent above those given in the manual. We base our figures on 1,500 cases, those in the manual are for 3,104 and 3,780 cases.

Classification by Grades or by Iowa Test Scores

We have used grades from the two preceding courses as a basis for classification. However, it seemed impossible to use such units of measurement as A, B, C, and D, grades for purposes of comparing at the first of the semester students from the experimental and students from the control groups. To say that all students with an A average or a B average from the control group had the same ability as those with the same A or B average from the experimental group was obviously inaccurate and unsatisfactory. A system of scores which might be divided into twenty or more groups lends itself to statistical treatment. For that reason the Iowa tests, rather than grades, were used. The sum of two perfect scores for the two tests is 140. Actually our scores ranged from 30 to 129.

The question arises as to the possible differences in results we would have found if we had used the Iowa tests to divide the students at a given hour into groups of three ability levels, instead of using the Iowa scores merely as the basis of statistical treatment of the data. A rather careful study of the scores of several classes shows that about one-fourth of the students would have been in a different section, either a higher or lower one, if the high section had included

approximately the upper third on the basis of the Iowa test, and if the middle section received the middle third, and the lowest section the low third. However, a further investigation of these students' individual records shows that of the one-fourth who would have been in a different section on the basis of the Iowa scores, about one-half of this group withdrew from the course before the end of the semester. The number of cases in which the difference of placement is significant is thus considerably reduced. The transfers made between sections took care of most of these remaining cases. The fact that there were so many withdrawals among students whose Iowa scores were at considerable variance with the previous semester's grades may lead one to consider favorably certain uses of our present maligned grading system. A study of the transfers made in all the sections for one year showed that about half of the time the transfer put a student into a section where his Iowa score, if used as a basis of sectioning, would have first put him. In the other cases the transfer put the student into a class where his Iowa score would not have placed him. In other words, transfers at the end of six and twelve weeks would have been as numerous under one system of sectioning as the other.

It would have been very difficult to have given the Iowa tests and then divided our groups into sections after a few days of school. Our students come from too many colleges (engineering, liberal arts, commerce, education) to make the matter of complete re-registration feasible for as few as six sections, even though some other basis of classification can be found which will be as satisfactory as classification by grades has proved to be.

Influence of Instructors

By using almost an entirely new group of instructors each year, fifty-five instructors have taught the sixty-six sections for which we have data. This change of instructors has led us to feel that the gain was independent of the teachers chosen. The results each year have been decidedly in favor of classification, although there have been certain instructors in both the control and experimental sections who were at least indifferent if not out of sympathy with the whole plan of experimentation. So many of the necessary features of the experiment, such as combined sections examinations, some supervision of instruction, and extra labor because of committee work,

are always unpopular on the college level that we have been very fortunate throughout the four-year period in having interest and cooperation from a large per cent of the instructors.

Conclusions

As for the results of the experiment, in the data on our main problem we found that a composite table for the four years gives positive evidence all along the scale that the classified sections made more progress in analytical geometry than did the control sections. The average by which the three-way classified sections exceeded the control sections for all groups of ability is 12 points for the combined June test scores. In the old type test the average advantage was 7.4, and in the new type test the average was 4.6. To understand the significance and the magnitude of these results, they should be compared with other significant measures of these data. Although the highest possible score in the June tests was 190 points, only in ten cases in the four years have scores exceeded 148 points. The group which made the highest score in the Iowa test averaged 148 in the June tests. We may then say that the gain of 12 points which the classified sections show over the control group is about one-twelfth this highest average score. In terms of the average of the whole four-year group of fourteen hundred students who took the June tests, we find that this average is approximately 92 points and that the 12 points gain is thirteen per cent of this average. If the gain of 12 points is to be interpreted in terms of the standard deviation and the frequency distribution, we find that the gain is equivalent to eleven per cent of the area under the frequency curve (*the graph shows this gain*).

The table shows clear evidence of the advantage of classification again in the two-way sections, but the gain is not of such magnitude for that group. Since the same system of regular transfers was not used in the two way sections, we are unwilling to say in any positive way that the three-way grouping is much better than the two-way grouping. We can only say that as we have conducted the experiment we have a much stronger case for grouping on the basis of three ability levels.

As a result of the rather careful and extensive study we have made, a report will soon be made by the chairman of freshman mathematics to the mathematics department. Certain recommendations as

to the future organization of courses on the basis of this report will be proposed at that time. Just what the action of the department may be in this matter is at this time a mere conjecture. Many of the details of the experimental procedure described need not be used except as an occasional check if the plan of classification is adopted. The whole experiment has been interesting and valuable from the point of view of those who have worked with it, and we are hopeful that there may be something of value in this statement of our conclusions to this larger group of mathematics teachers.

Notice to Council Members

The National Council of Teachers of Mathematics is very desirous of increasing its membership to 10,000. This goal can easily be accomplished if every member will do his part. Thousands of advertising circulars and many copies of THE MATHEMATICS TEACHER have already been sent to members in strategic places all over the United States and abroad. Obviously there are other members who are in a position to help. If any member knows of a meeting of mathematics teachers that is to be held in his locality during the year The National Council will be grateful if such persons will write to THE MATHEMATICS TEACHER stating the number of advertising circulars desired and the number of sample copies of the magazine that can be used to advantage.—THE EDITOR

Solid Geometry in the High School

By Dr. A. B. Coble
University of Illinois

The present situation of solid geometry in the high school, and current tendencies with respect to it, are discussed in a number of articles in the 1930 Year Book of the National Council of Teachers of Mathematics. W. D. Reeve of the Teachers College in Columbia University says in an article entitled "The Teaching of Geometry": "The pressure upon the curriculum, the new subjects that are clamoring for a place in the sun, and the demands that we break with tradition—all these problems have caused educators to question the extent to which geometry should be carried. The wisdom of teaching solid geometry in particular has been seriously questioned. In fact solid geometry as a separate half-year course is rapidly becoming passé in our schools. It is not even required for entrance in some of our engineering schools and colleges, as, for example, the following statement shows":

The statement then presented emanates from the University of Nebraska. No other instances are given.

Reeve remarks later that "if geometry is to be preserved, it can be done in three ways.

"(1) It can be taught as a separate unit in the tenth grade much as it is done today except that it must be more concentrated.

"(2) It can be fused with plane geometry at the places where such fusion seems most desirable and possible.

"(3) It can be taught intuitively."

W. R. Longley of Yale University, under the title, "What Shall We Teach in Geometry," stresses the growing demand for mathematical training. One of his examples is the following: "A short time ago a portion of the staff of the Medical School of Johns Hopkins formed a class and requested a member of the department of mathematics to give them a course in the calculus." To this I may add that, as long as fifteen years ago, it was understood that undergraduates applying for admission to the Johns Hopkins Medical School would receive more favorable consideration if their courses

included the calculus. Longley advises the retention of solid geometry with the adoption of an intuitional and practical, rather than a logical approach to the subject. He mentions an interesting situation in Minnesota. The faculty of the University of Minnesota eliminated solid geometry from the university curriculum presumably on the ground that it is a high school subject. The high schools of the state offered only one year of plane geometry and probably would not offer more geometry. Thus solid geometry becomes a sorry orphan, indeed. The high schools refuse to mother it and the university refuses to father it. Under these distressing circumstances Dunham Jackson proposes a unified course through one year.

May L. Wilt of the University High School of West Virginia University, under the heading, "Teaching plane and solid geometry simultaneously," speaks of this unification as an accomplished fact. She says: "Under the pressure of change in secondary mathematics plane and solid geometry have been compressed into a one-year course. Such a combination entails two marked changes: first, the elimination of about one-third of the content, and second, the simultaneous introduction of the more fundamental parts of plane and solid geometry."

A questionnaire recently circulated by Chas. A. Stone as a basis for discussion at a meeting of the Central Association in Milwaukee opens with this question: startling, indeed, to one who has not followed recent developments in the high schools: "Do you feel that solid geometry should be taught in the high school?"

It is thus quite clear that the traditional year and a half of high school geometry is in serious danger of curtailment and that teachers are facing the adjustment to such a change. No one, least of all the teachers of mathematics, would seriously contend that every student should take all, or perhaps even any, of the mathematics offered in the high school. They would much prefer to deal with students who elect the subject because it excites their interest or serves their further purposes. That the high school should deny to these students the opportunity to satisfy their needs by refusing to offer certain courses is clearly a dubious and debatable proposition. Such a proceeding could be justified only on one of three grounds: (1) that the content of the course is not of sufficient intrinsic importance, in comparison with courses that might replace it, to justify its retention; (2) that properly prepared students do not profit adequately from the

presentation of the material; (3) that not enough students elect the course to form a class.

This last condition should be regarded as merely an administrative exigency and should not bar further offering unless indeed the condition become chronic. Even then, if the course could be justified on the first two counts, the occurrence of this last situation should be regarded as evidence of intellectual apathy in the student body which teachers should labor to dispel.

That the relative importance of mathematics in the general scheme of human knowledge is increasing rather than diminishing can hardly be denied. As knowledge in one field after another tends to become more exact, so also does it tend to seek expression in mathematical form. The ability to comprehend this form of expression becomes more and more important to the individual. Thus the obligation of the schools to foster this comprehension becomes more compelling. Only a minority of the students may be expected to contribute to the intellectual advancement of their fellow-men. An individual school may not in centuries produce an Einstein to construct a new model of the universe or a Bohr to construct a new model of the atom. Yet the school should, from its very nature, seek to keep open the path to the highest scientific and intellectual achievement.

In mathematical science geometry occupies a place quite as important as that of analysis and algebra. It is in the interaction between these fields that many advances in the subject have been made. A long line of famous mathematicians—Riemann, Abel, Klein, Lie, and Poincaré among others—have shown the importance of geometric concepts in enlarging the domains of algebra and analysis. The theory of point sets, fundamental in analysis, and of analysis situs, fundamental in geometry, are so closely allied that no line of separation can be drawn. In modern theories of chemistry and physics also, quantitative concepts are being replaced by structural relations. A knowledge of geometry is essential for the proper understanding of these sciences.

Of all forms of geometry, solid geometry as taught in the high schools has more applications in practical every-day life than any other. No one complains when the high schools require four years of English. The importance of the spoken and written word justifies it. But surely, in comparison, a half-year devoted to the study of the space in which we live and move and have our being is not excessive.

If you will agree with me, at least for the moment, that solid geometry has sufficient intrinsic importance to justify its present—perhaps I should say its late—position in the high school curriculum, let us consider the question to what extent students profit from their study of geometry. In this matter it has always seemed to me that there is a notable difference between plane and solid geometry. Our incoming freshmen who elect mathematics are probably an average group of high school graduates. To counterbalance those who are fairly expert in mathematics we get those who merely prefer it over something worse such as physics or chemistry. The major portion of this group retain, after a lapse of several years, sufficient command of the fundamental concepts, principles, and facts of plane geometry to progress naturally into graphical algebra, trigonometry, and plane analytic geometry. It seems only a fair inference that plane geometry is well and profitably taught in the high schools. On the other hand even the selected group of mathematics majors seems to experience great difficulty with the concepts of solid geometry which are involved in the solid analytic geometry and certain topics in the calculus of the sophomore year. Ordinarily about half of this group have had solid geometry in the high school, and all have had the benefit, if any, that may be obtained from that pitiful little chapter on solid analytics which concludes current texts. Sections of engineering students go through the calculus far more smoothly. But an engineering faculty is composed of realists who, wanting students with a knowledge of geometry, first require them to take all that is given in the high school, and then administer a stiff prescription of mechanical drawing and descriptive geometry. Students sometimes complain that this dose is bitter but not that it is ineffective.

Two reasons might be advanced for this unsatisfactory "carry over" in solid geometry: first, that solid geometry is far more complex than plane geometry; and second, that not enough time is allowed to digest the material when this is presented by the logical method. Miss Wilt in the article quoted earlier remarks: "Spatial concepts and plane concepts contribute much to each other, but since we live in a three-dimensional world the spatial concepts should be easier of comprehension." It is true that the student is able to move about in the world with reasonable safety. He may even have learned to do some things which require rather delicate spatial adjustments, as in playing pool. But this type of knowledge is only mildly helpful

in a systematic analysis and classification of space figures and their properties. Moreover his previous intellectual experiences have centered largely about the plane of the paper and the plane of the blackboard. It is much more difficult to perceive, even with the aid of models, that the eight corners of a cube divide in just one way into two regular tetrahedra than that the six corners of a regular hexagon divide in just one way into two equilateral triangles.

The importance of the element of time in the comprehension of mathematical ideas is familiar to every one who has mastered any branch of mathematics. The individual student can obtain this comprehension only as the result of his own thinking. The end can be facilitated in some degree by judicious choice and skillful presentation of the material of the subject. But it is by the reiterated perception of fundamental concepts and their relations in a variety of situations that the student gains some appreciation of their validity and incorporates them into his own intellectual background. In solid geometry some time can be gained for practice with fundamentals by omitting the more technical part of the subject matter and by dropping the logical method wherever intuitive methods form a satisfactory substitute.

The general adoption of a unified course in plane and solid geometry running through the year would be in my opinion most unfortunate. The unification of knowledge is a secondary process which occurs naturally after various concrete bodies of knowledge are so thoroughly known that their abstract qualities can be compared. A unification which to the trained teacher may seem quite simple and easy may appear quite artificial to the student. If however unification means merely the simultaneous presentation of related ideas in the two subjects then the student is likely to become quite lost in the multiplicity of new concepts. If unification means the successive presentation of the two subjects, each for part of the year, then it is very likely that the students' knowledge of each will be so shallow that the teacher will be forced to stress the one at the expense of the other, the other being solid geometry.

I trust that the members of this body will do all that is in their power to retain solid geometry as an elective in the high school curriculum. With proper choice of material and method of presentation the subject should attract a fair proportion of the better students. Its abandonment would indicate a distinct lowering of the level of intellectual opportunity afforded by the school.

Some Methods in Professionalized Subject Matter Courses in Mathematics for Teachers College

By EDNA E. KRAMER
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AS A RESULT of the recent interest and progress in teacher-training in the United States, has come the evolution of the *normal school* into the *teachers college*. The development has naturally given rise to the question of what should be done in order that the lengthened course be filled in most profitably. Whether to give additional courses in educational theory and methods of teaching, or to include courses in the content of the various subjects which students plan to teach—that is, whether to make the teachers college a normal school of a "larger growth," or to convert it into the equivalent of a liberal arts college, which would lay special stress on the subjects commonly grouped under the heading "Education"—these have been the points under consideration.

The consensus of opinion has it that neither of these policies should be pursued, but, instead, something called "Professionalized Subject Matter" is to be taught—which is to be neither *method* nor *content*, but both, and *more* than both.

It is the purpose of this article to list some ideas which, in the opinion of the author, might be utilized in "Professionalized Subject Matter" courses in mathematics, given at teachers colleges.

These points are classified under the headings of (1) *Imitation*, (2) *Creativeness*, (3) *Special Topics*, (4) *Miscellaneous Possibilities*.

(1) *Imitation*. It is a well-known fact that students imitate their teachers, and, to a high degree. For this reason, inexperienced teachers, who present a topic for the first time, unconsciously teach it as they have been taught.

Let anyone who doubts this fact hold a "socialized recitation," or, in some other way, have students conduct the business of instruction, while he himself remains a silent observer. He will have occasion to be amused, perhaps even embarrassed, by the students' imitation of him—even to the use of his mannerisms, and this without malice aforethought.

Thus it is obvious that the teacher of teachers must be most cautious in his own choice of method.

In this connection, *his own teaching should reveal modern trends.* Although he may be dealing with material in higher mathematics which his students may never have occasion to present, he is developing in them a predilection for certain types of work.

In one breath, he may tell his classes that thought, and not manipulative skill is the 1931 style in the teaching of algebra, and in the next, assign the following example for consideration:

Find the derivative of $y=\left(\frac{x}{1+\sqrt{1-x^2}}\right)^n$.

His students, with their absolute faith in him, will fail to see the inconsistency, but, instead, with due conscientiousness, will develop a real fondness for this sort of thing, and, in two or three years will not consider themselves unreasonable in demanding that their pupils solve:

$$\sqrt[3]{8x+4} - \sqrt[3]{8x-4}=2.$$

When the subject matter of high school courses is handled, it should be presented at some time just as it would be done in a high school class. This may be done initially, or after the subject has been treated from the more advanced point of view. For example, after a study of maxima and minima in the differential calculus, the student expresses astonishment when told that such work can be included in algebra courses, perhaps as early as the ninth year. A brief discussion of the graphic treatment of such problems convinces him readily, and he is interested in obtaining some practice in working examples from the more elementary point of view.

In reviewing the topic of logarithms in the freshman course in college algebra, the author has found it possible to correct hazy notions as to theory, and, at the same time, to give helpful professional information, by giving as the initial treatment the method followed in "High School Mathematics" by John A. Swenson. At first some students consider themselves superior to this elementary procedure, but eventually all of them feel that they have acquired knowledge which will be an important aid in their teaching.

A *preliminary survey course* may serve a useful professional purpose.

Professor Nordgaard, in his article in the *Third Yearbook of the*

National Council of Teachers of Mathematics, on "Introductory Calculus as a High School Subject," has very aptly described the attitude of the average secondary school teacher:

> Let the question of teaching the calculus in the high school be raised, and in the mind of some teacher is conjured up a picture of the calculus class in the sophomore or perhaps the senior year of college, where a learned professor expounded to the intellectual elite of the college the Mean Value Theorem, singular points, or moment of inertia. Such a teacher looks at the 17-year-old boys and girls of his high school and cannot imagine these youngsters taking the calculus.

Hence it behooves the teachers college instructor to make the prospective teacher feel that there is nothing stupendous about presenting the subject to high school pupils. Although it is customary to devote the first month or two of the usual college course in the calculus to the acquisition of the technique of differentiation, it might be well, in a teachers college, to leave the beaten track, and devote this time instead, to a brief survey of the calculus in just the way that it would be handled in the twelfth year course proposed by the National Committee on Mathematical Requirements, that is, the calculus of the algebraic polynomial could be made the subject of preliminary consideration.*

Such a study of the calculus makes for greater interest, in addition to serving the purpose of professionalized subject matter, since the student acquires the fundamental notions at the outset, and avoids what is a common occurrence among students of the calculus—that of not being able to see the basic ideas through the haze of technique with which they are surrounded.

Because the instinct of imitation is so strong, it would seem advisable to warn students as to what they ought *not* to teach as they have been taught. An enthusiastic student-teacher once told me of her disappointment at her lack of success in explaining the factor theorem to a pupil whom she was tutoring in high school algebra. When I questioned her as to method, she told me that she had first developed the remainder theorem, and had then taken the factor theorem as a corollary, just as she had studied the facts in college algebra, since this seemed to be the most logical course. This illustration indicates that it might be well, very often, as in the case of

* This preliminary treatment could include the material given in J. A. Swenson "Selected Topics in Calculus for the High School," found in the *Third Yearbook of the National Council of Teachers of Mathematics,* and other topics as well.

maxima and minima mentioned above, to pause and ask, "How would you teach this to a high school class?"

Students who have heard reiterated that modern principle in mathematics instruction, *"Stress the function concept"* need to be saddened with the information that those facts from the theory of functions which seemed so elucidating and interesting in their advanced calculus course, cannot be transferred as such to their high school pupils.

Since students tend to imitate our strong points as well as our defects, it would hardly be a breach in the matter of modesty to *call attention to pedagogic devices* which we may employ. Having developed the formulas for the derivatives of sine and cosine, we almost unconsciously select practice examples in the following order:

$$Y = 3a \cos X$$
$$Y = 3a \sin 2X$$
$$Y = \sin 2X \cos X$$
$$Y = \sin^3 X$$
$$Y = \cos \sqrt{x^2 - a^2}$$
$$Y = \sin^2 \frac{1}{\sqrt{x^2 - a^2}}.$$

The motion of a body is given by the equation $X = a \cos kt$. What is its initial velocity? Find also the initial acceleration. When is the velocity a maximum?

It might be worth while to call attention to the reason for this particular choice, and ask students to prepare a short list of exercises arranged in the order of their difficulty for some topic in elementary algebra—factoring the difference of two squares, for example, or from geometry—as applications of the Pythagorean theorem.

The *use of texts* is another matter in which we may exert a strong influence upon our students. Teachers who have learned their mathematics by the textbook method—that is, by the use of a single text in each course, studied thoroughly from cover to cover, by taking all topics in the order prescribed by the author of the text, will be likely to teach their pupils in the same way. They will not see the advantages in omission and rearrangement of work in a text, nor feel the need of supplementing it. Nor will they fall readily into experimenting with ideas which do not occur in any text. Hence, all the last-mentioned procedures should be carried out in mathematics courses for teachers. Students should have assignments from several texts, and from sources, and should have the opportunity to see variety in proofs and in sequence of topics.

Creativeness. Whether creative power can be cultivated or de-

veloped to any considerable degree is questionable. Yet a certain amount of originality is certainly desirable for a teacher of mathematics who expects to hold the interest of pupils year after year with the immutable facts of the science. And surely a greater degree of creative power ought to be fostered in those who have the ability and the desire to be leaders in the profession. Even if the ability to think along novel lines is innate, training can nurture or stifle it.

The "original" is a fetish in mathematics, and it seems to be the opinion of many, that ability in these exercises in deduction has a close connection with originality in thought. This would be so, if those who solved the problems had also set them. I should venture a guess that the correlation between ability to solve "originals" and to compose thought-provoking mathematics questions would not be high.

All student-teachers are potential textbook writers, or, at worst, prospective composers of examinations. And hence it would be worth while to give them a little training in the art of obtaining variety in the setting of problems. In this connection they may learn how to *magnify* the difficulty of too straight-forward a problem, or how to *diminish* the abstractness of a question beyond the students' level. The following is an illustration of the former process:

When students have worked one or two examples of the type:

For what values of k will the straight line $y = 2x + k$ *be tangent to the circle* $x^2 + y^2 = 49$?

they may be asked to compose examples which involve the same technique, but introduce some additional difficulty (and to do this without recourse to any text). This procedure was followed in an analytic geometry class at Montclair and the following problems were readily produced:

Find the equations of the straight lines of slope 2 tangent to the circle $x^2 + y^2 = 49$.

Find the equations of the straight lines parallel to $2y = 4x + 3$ *and tangent to the circle* $x^2 + y^2 = 49$.

Find the equations of the straight lines perpendicular to $2x + 4y + 1 = 0$ *and tangent to the circle* $x^2 + y^2 = 49$.

Find the equations of the straight lines tangent to the circle $x^2 + y^2 = 49$ and *having an inclination of 135°.*

Find the equations of the straight lines tangent to the circle $x^2 + y^2 = 49$ and *making an angle of 60° with the line* $x = y$.

Find the equations of the straight lines of slope m tangent to the circle $x^2 + y^2 = r^2$.

Find the area of the triangle formed with the axes by either of the tangents of slope ½ to the circle $x^2 + y^2 = 49$.

As an illustration of diminishing the difficulty of a problem, they can readily learn such simple devices as converting problem (1) below, into problem (2) for the purpose of making it easily intelligible to high school students.

(1) *Find the most economical proportions for a rectangular box with an open top and square base.*

(2) A rectangular box with open top and square base is to have a volume of 108 cubic inches. Find what its dimensions should be if a minimum amount of material is to be used in its construction.

The *keeping of a notebook* in a mathematics course can also serve the purpose of stimulating original thought. To be sure, if the notebook is an exact transcript of lectures, or a record of textbook examples, it will have no such value. But lecture notes can be reorganized, and amplified by graphs, notes on outside reading, original illustrations and comments. The most interesting and the most varied illustrations of the distinction between a "necessary" and a "sufficient" condition which I have ever read, have been those in the notebooks of Montclair students. The student who gives $\sin x + e^{-\frac{1}{x_2}}$ as an illustration of non-analyticity, instead of $e^{\frac{1}{x^2}}$ given in the lecture, is not necessarily a budding mathematician, but at least he is headed in the right direction.

Special Topics. One of the things which makes the lot of the college instructor a happier one than that of the secondary school teacher is the fact that his students have no extra-mural examinations to face, and he has no college entrance requirements or regents' syllabus to meet. On the other hand, it places a very grave responsibility with him. He must choose from the large amount of material available, all worth while from the mathematical point of view, those things which will be most profitable for his students. The teachers college instructor has an advantage over other college instructors in this matter. The needs of his students are more nearly uniform. Liberal arts colleges group their students into "pre-engineering," "pre-medical," "pre-law," "pre-journalism," etc. sets as far as requirements and advice as to selection of courses go, but cannot differentiate them in the college classes themselves, so that a mathematics course must meet many divergent needs.

Obviously, there are certain mathematics courses, as, for example, History of Mathematics and Statistics, which should be taken by all who are preparing to teach, but even in the other courses, *certain top-*

ics should receive particular stress, because they are being studied by those who are to teach high school mathematics.

Although it has become quite fashionable to fuse analytic geometry and calculus with very much less emphasis on the former, and it is really an economy of time to do so, there are nevertheless certain important things for secondary school teachers to gain from a detailed study of certain ideas in cartesian geometry. First, the fundamental idea which is likely to be given all too brief a treatment in a combined course—the *locus concept.* The teacher should become acquainted with it in all its grandeur, lest he be lost in later years in the diminutive locus questions of secondary school geometry texts.

Since the *conics* furnish important material in the high school work on the graph, the teacher should be sufficiently well-informed as to their nature, on the general principle that the teacher should be more than one step ahead of his pupil. He should know, for example, some of their interesting properties, and be able to recognize the species of conic from the general equation.

For the same reason, *curve-tracing* is a topic of the calculus which should be dwelled upon.

As another illustration we may mention the topic of *infinite* series. It should be handled with special emphasis on the computation of logarithms and trigonometric functions. While the teacher never shows his pupils exactly how the tables they use are derived, this is another point on which he should have the advantage of superior knowledge.

Miscellaneous Points. There are, without doubt, countless other devices for professionalization in a variety of different situations. We give as a final illustration the question of review. Many occasions for such a procedure naturally arise. For example, in the college algebra course it is really essential to have a preliminary review of some of the earlier principles. The heterogeneity in earlier training, in memory and ability of pupils can here serve a professional purpose. Those students who are at an advantage and who may feel that repetition is unnecessary can be called upon to present the subject matter. Their frequent inability to do so in a concise manner, soon convinces them that the knowledge which they thought complete, must have gaps in it, or else that there is a little more to the teacher's art than the mere cognizance of facts.

A number of additional devices might be here described but, since the above illustrations are typical, further details would be superfluous.

Some Simple Logical Notions Encountered in Elementary Mathematics

II

ON PROVING IDENTITIES AND SOLVING EQUATIONS[7]

By CARL A. GARABEDIAN
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IT IS A serious shortcoming of our elementary texts that they fail to explain the logical notions which underlie the processes employed in proving identities and in solving equations. The situation might be less embarrassing were it not met so early. But it is at the very beginning of the study of formal algebra that it becomes necessary to distinguish between, and to deal with, identical equations and conditional equations; and, following closely upon the course in algebra, comes trigonometry with its chapter on identities.

Employing the very useful symbol $\equiv$ to distinguish an identity from an equation, let us remind ourselves at the outset that equations, whether algebraic or trigonometric, are conveniently classified under two headings:

(I) *Identical Equations.*—An identical equation (more briefly, an identity) is one which is satisfied by all values of the variable (or variables) for which the equation has meaning. For example,

$$x^2 - y^2 \equiv (x + y)(x - y)$$

is an algebraic identity—as are also, of course, further special products listed in the chapter on factoring in the usual algebra text; $\tan\theta \operatorname{ctn}\theta \equiv 1$ and $\sin^2 x + \cos^2 x \equiv 1$ are examples of trigonometric identities.

(II) *Conditional Equations.*—A conditional equation (more briefly, an equation) is one which is satisfied by only particular values of the variable (or variables) involved. For example, $x + y = 2$, $x^2 - y^2 = 25$ are algebraic equations; $\sin x + \cos x = 1$, $\sin\theta \cos\theta = 0$ are trigonometric equations.

We hasten to add some remarks which, by centering attention on

[7] The first paper, entitled "Necessary, sufficient, and necessary and sufficient conditions," appeared in the preceding issue of this journal.

exceptional cases that may arise, serve to make clearer the distinction just drawn. Note that

(a) An equation may have no solution whatever (see the end of this paper). On the other hand, it may have an infinite number of solutions—as does, for example, the trigonometric equation $\sin x = 0$. An algebraic equation in one unknown can have only a finite number of solutions.

(b) The equation $\sin x = 0$, although it has an infinite number of solutions, is not an identity—since, obviously, not all angles reduce the left-hand member to zero.

We turn first to the topic of trigonometric identities, since it is here that we find most of the difficulties and misunderstandings.

In connection with the proof of a trigonometric identity, there is perhaps no point more deserving of emphasis than this: that there is no such thing as *the* method of proof. Not only is there a wide choice of method; but, what is more, when once we have decided on a method that is logically correct, there may remain a number of different modes of procedure. If we think that there is but one correct method of proof, we err as does the scientist who fastens upon some particular mathematical theory as the *only* explanation.

A good illustration of the point we are now stressing is afforded by that method of proof in which we transform to an algebraic identity by substituting, for the trigonometric functions, the original ratios taken from the triangle of reference. This method of proof is open to objections pedagogically, for it frequently involves difficult algebraic transformations and awkward questions of signs and it dodges completely the trigonometric transformations. But it is a perfectly correct method of proof, and there are occasions in mathematics where this type of proof may be employed to advantage. Is it not, indeed, the type of proof we naturally use to establish the Pythagorean relations $\sin^2 \theta + \cos^2 \theta \equiv 1$, $1 + \tan^2 \theta \equiv \sec^2 \theta$, $1 + \operatorname{ctn}^2 \theta \equiv \csc^2 \theta$? Moreover, this being the first method used in the text, is it surprising that the student should consider it anomalous that we proceed forthwith to forbid him the use of the method?

There are also the methods of proof which make use of the line-value representation or the graph. At times it is precisely one of these types of proof which takes first place in choice of method. Naturally, teachers prefer, in most cases, to make a proof by means of the fundamental identities. And why? For the simple reason that

this method gives the student a training in manipulation of fundamental formulas—a technique not secured by other methods. But there are virtues in some methods of proof which are not accorded conspicuous classroom use. And is it not questionable instruction which leaves the student with the impression that there is but one correct method of proof?

The present paper is concerned mainly with the method of proof by means of the fundamental identities. Even in this method we find a number of possible modes of procedure—and, still more embarrassing—we are confronted by various forms of presentation and ways of arranging the work on the page. As to modes of procedure, we may

(1) Start with a known identity and, by means of the fundamental identities, and the processes of elementary algebra, derive from it the new identity.

(2) Assume the given expressions identically equivalent and, by operating on both sides (with operations legitimate because of the initial assumption) deduce a known identity. This procedure, and how it may be turned to yield a proof, will be discussed shortly.

(3) Assume no equality between the two given expressions to be proved identical, and use the fundamental identities to transform, step by step, either (i) one expression into the other, or (ii) both expressions into the same expression.

We might, on occasion, prove two expressions identical by showing that their difference is identically zero, but this would be merely a variation on procedure (2) or procedure (3).

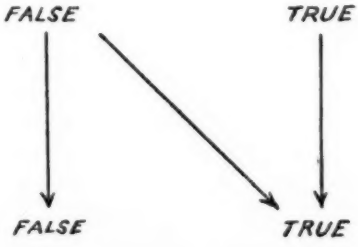
We use method (1) in deriving new identities—and, incidentally, to complete the proof in method (2). But (2) and (3) are the methods useful in dealing with identities proposed for verification; in short, these are the methods which the student will use to solve the conventional type of drill problem. We shall find it illuminating to scrutinize closely the logic underlying these two methods.

In method (2) the *proof* consists in showing that it is possible, by starting with the known identity and retracing the steps which have "blazed the trail" for us, to get back again to the given identity.[8] But apparently this is not a widely-known fact. About ten years ago,

[8] For a proof, it is the steps, not the operations employed, which must be uniquely reversible. Observe that a step may be non-reversible because it is not reversible at all or because it is reversible in more than one way. The manner of reversal obviously need not be unique; it is essential only that we be able to get back, by valid steps, to the point of departure.

I examined, on this point, all the trigonometries I could lay hands on, and found only two books which even attempted to explain the situation. It is certainly not by means of the textbook that knowledge of these matters has been disseminated. Moreover, it has recently come to my attention that certain teachers are still shunning this method as if there were, after all, something wrong about it. There can be no harm, therefore, in venturing a little further with our explanation.

For the *bone fide* examples of practice, method (2) is doubtless the most tractable method of all. It finds constant application, not only in proving identities, but also in dealing with mathematical problems generally. It is, so to speak, the method of Sherlock Holmes: the method which proceeds from some clue, step by step, to a fact which, provided steps are retraceable, may discover the criminal. By all means, then, let us familiarize the student with the fact that from a false premise perfectly valid steps may lead to either a true or false result; that from a true premise, however, correct steps can lead only to a true result. That is, schematically:



Observe that there is no path, by valid steps, from the true to the false. But, of course, by introducing a false step, we may, even with a true premise, arrive at a false conclusion.[9]

Since, in identities, it is virtually only the squaring operation—seldom multiplication by zero—which can raise the question of reversibility, there is really no good reason for rejecting method (2); and, as we shall see, it is this method which is universally employed in solving equations. To slur over the method in the chapter on identities is both to disregard a very useful tool and to deprive the student of vitally important information in logical methods. As already pointed out, our chief reason for putting method (3) ahead of method

[9] For examples, see Garabedian and Winston's *Plane Trigonometry*, McGraw-Hill, 1929, where are elaborated many points here treated only briefly.

(2) is one that concerns technique in transformations. But although we may require the student to make his proof by method (3), let us not neglect method (2), since, by a discussion of various methods of proof, we give the student something which may stand him in better stead than special technique. Moreover, for many examples that arise in practice, method (3) may be found lengthy and it may not be obvious what steps to take; we then do well to turn to method (2)—as does so frequently the professional mathematician when he is solving a problem of his own.

We come finally to method (3), and to an occasion for, perhaps, the most important of all our remarks. Here again we take especial interest in the method *per se*, since it is a new experience for the student to make a proof by means of identical transformations. The logic is simple, but the novelty in the reasoning may bring certain difficulties. For example, some teachers seem still unsettled as to whether $A \equiv B$ is the same thing as $B \equiv A$; some feel that, having transformed one of the given expressions into the other, the proof is not complete until it is shown that the second expression is transformable into the first. Perhaps the confusion here springs from the desire to treat the relationship of identity as if it were one of implication—or is it that $\tan \theta \operatorname{ctn} \theta \equiv 1$ seems more reasonable than $1 \equiv \tan \theta \operatorname{ctn} \theta$? Then there is the tradition which makes it sinful to "work on both sides." The fact is that, given two expressions equally involved, or possessing certain symmetry, working with both sides is a procedure to be recommended. Furthermore, to arrange the work so that only one side appears as transformed may seemingly secure elegance, but only at the expense of changing—by reversing the natural order of steps—a simple proof into a difficult one. It would be more to the point to seek elegance in a better choice of transformations.

Of course method (3) furnishes a direct proof (with no necessity of retracing steps) as soon as both the given expressions have been reduced to the same expression. But we may now employ, since no equality of the expressions is assumed at any stage of the proof, only operations applicable to one side at a time; that is, we may (i) expand and multiply out, (ii) collect terms and factor, (iii) combine fractions and reduce to lowest terms, (iv) add and subtract the same quantity, (v) multiply numerator and denominator of a fraction by the same quantity, and, naturally, (vi) substitute for any expression another known to be identically equivalent.

A student may work his proof by method (3), taking only steps of the type just listed, and yet be unjustly accused of using method (2). For instance, a student who is asked to show that

$$\tan x + \operatorname{ctn} x - 2 \csc 2x \equiv 0,$$

writes on his paper:

(1) $$\tan x + \operatorname{ctn} x - 2 \csc 2x \equiv 0$$

(2) $$\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} - \frac{2}{\sin 2x} \equiv 0$$

(3) $$\frac{\sin^2 x + \cos^2 x}{\cos x \sin x} - \frac{2}{2 \sin x \cos x} \equiv 0$$

(4) $$\frac{1}{\sin x \cos x} - \frac{1}{\sin x \cos x} \equiv 0$$

(5) $$0 \equiv 0$$

There can be no more certain way of starting an argument than to show this proof to a group of teachers. At once flows a torrent of protest: But he starts with what he was asked to prove! He thinks he is proving that $0 \equiv 0$! He must be using method (2), for, see, he writes "$\equiv 0$" at each step! And so on!

The objections just raised are of the sort read into the proof by one who is prejudiced by the form adopted by the student. Obviously the left-hand member has been reduced, by identical transformations, to the right-hand member; and he who maintains that the student assumes, at each step, the truth of the preceding identity, would doubtless find it disconcerting to be asked to reproduce the student's faulty reasoning. That the proof is a good one is readily seen by inserting, between each of the numbered identities, some such phrase as "which is the same as asking me to prove that," just as Curtiss and Moulton, on page 104 of their *Trigonometry*, very happily supply the phrase "which is true if" to connect the steps in a proof by method (2). As to connecting phrases, the student need not clutter his proof with explanatory prose; he may arrange his work schematically, as in solving any equation. It is not a case here, as in plane geometry, of different arguments at each stage; the logical argument is now the same at each step and is, therefore, advisedly omitted. There can be no objection to the use of some appropriate schema; the important thing, after all, is that both student and teacher should understand the reasoning behind the schema.

Many teachers attempt to impress upon the student the fact that he is no longer using method (2) by asking him to rule a vertical line between the two expressions to be proved identical; others go further and insist upon a separate handling, in the style of running prose, of first the left-hand member and then the right-hand member. Whatever the form favored by the individual teacher, the student must not be left with the impression that he is using *the* correct procedure. For it is dangerous to know *the* right way of doing a thing; better far to be able to recognize *a* way when one appears on the scene.

There remain points of logic we have not touched upon at all. The student may be asked to find, in each problem, the angles for which the identity has no meaning. His attention may be called to the fact that no matter how many particular angles are shown by actual test to satisfy the relation, this alone can never furnish a proof; that it takes only one angle, however, to prove that a relation is not an identity, namely, an angle which gives meaning to, but does not satisfy, the relation.

Our remarks concerning the solving of equations can be made briefly. We would chiefly stress the importance of explaining to the student the logic involved in the process of solving an equation—a matter fully as deserving of study as the extraneous root and the lost root. The logic is this: we always assume that the equation has at least one root and then operate on the equation by valid algebraic processes until we arrive at what must be the root, or roots, *provided there be any*. Finally we check, by substitution in the given equation, to discover if our tentative roots really satisfy; occasionally, as in the quadratic, we check, once for all, in a general fashion. But this whole situation is often lost sight of because of the fact that ordinarily the answer does check.

Much light can be shed on what is actually going on by giving the student an equation like

$$y + \sqrt{y^2 + 7} = -2,$$

which has no root. The usual procedure leads to $y = \frac{3}{4}$ as the root, *provided there is one*. By this example,[10] we may demonstrate to the

[10] The example need not contain a radical; we might have written, for instance, the equation:

$$\frac{2x}{x^2 - 9} - \frac{1}{x - 3} = 0.$$

student the limitations on the Fundamental Theorem of Algebra, and show him that failure to check does not necessarily mean a mistake in the work. We may even lead him to see the importance of understanding the logic behind the formal procedure.

The logical notions elaborated in the present paper—like those considered in Paper I—have yet to be explained in our textbooks. Here is a clear case where the teacher, by supplementing the material of the text, can render the instruction less formal and uninspiring.

The Rhind Mathematical Papyrus

It is to be hoped that members of The National Council of Teachers of Mathematics will give particular attention to the review of Chancelor Chace's book, "The Rhind Mathematical Papyrus" on page 460 of this issue. This is a valuable publication for teachers of mathematics and should be on the shelves of every high school library.—THE EDITOR

Constructing a Transit as a Project in Geometry

By T. L. ENGLE, *Drawing by* MAX CONNELLY (Student)
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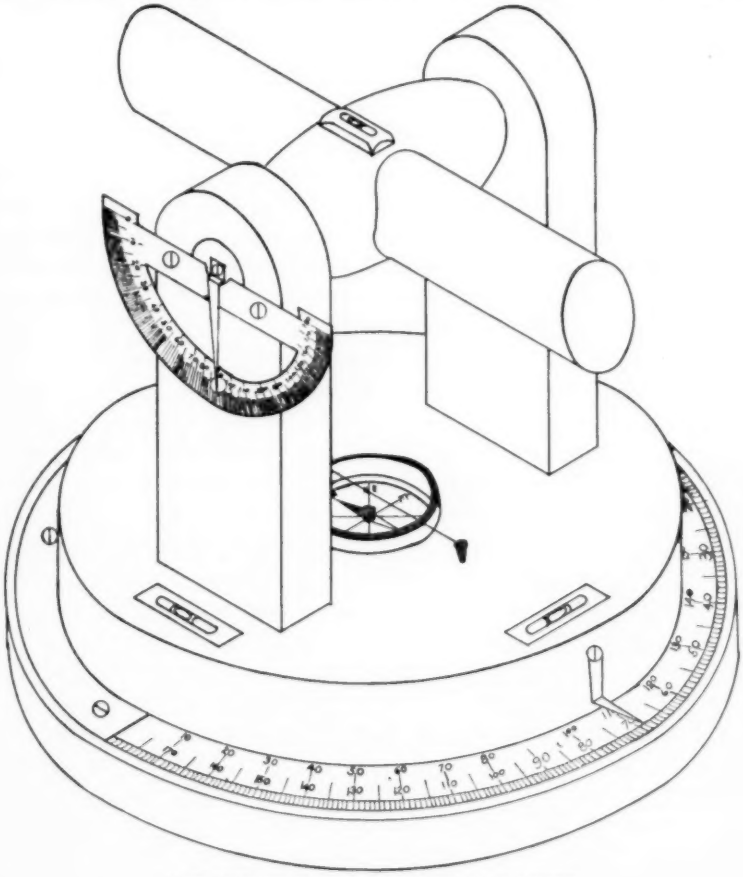
PSYCHOLOGISTS may call it by various names, but teachers of mathematics know that most boys pass through a stage of development which might be called the "civil engineering" period. Fortunately for the engineering profession and for the individuals, the majority change to other fields. However, during this period of development, the interest in engineering is real and it may be used to advantage by the high school teachers of mathematics, especially in the geometry courses.

Our colleagues in education, the leaders of the Boy Scouts, have recognized this urge more than we have, and we can motivate our work by building upon the foundation which they have laid. Probably the boy has passed his first-class map making test by the time he begins his high school work in geometry, but about this time he is likely to be very much interested in merit badges, especially in the "Surveying" badge. The *Handbook For Boys* and the merit badge books on "Surveying" make excellent supplementary texts for geometry. Furthermore, First Class Scouts who are working for this badge can furnish much information and skill that will help to motivate the course during those first few weeks when a teacher must break down the prejudices against geometry which have been handed down from school generation to school generation.

The mere presence of a transit in a mathematics room on the first day of the school year may develop interests which will last throughout the course, and possibly through life. However, professional transits are seldom found in high school classrooms. For one reason, the cost is prohibitive in most school systems. Furthermore, they are so complicated that the student may be overawed by the technical points involved. Even though a good instrument may be provided its cost and delicate mechanisms may prevent its actual use in the field by students. Students do not wish to look at a transit; they wish to use it, and some may wish to borrow it for map making in connection with

scouting activities. Moreover, teachers would hardly feel justified in loaning a professional instrument to the average high school student. The author has developed a classroom project which will provide an instrument costing not more than $2.00, one which will be simple



A TRANSIT FOR GEOMETRY CLASSES

enough for any student to understand and use, and also one which is not too delicate for student use.*

An isometric drawing has been used in this description rather than detailed drawings inasmuch as it is desired merely to outline a general

*A leveling rod will provide another project and will add interest, but it is not within the scope of this article.

plan. Without doubt any teacher caring to work out this project will find details which can be improved. The only materials that need to be purchased are two protractors (one 8″ in diameter, the other 4″ in diameter), a pocket compass (1½″ in diameter), and three small unmounted spirit levels. The remainder of the material can be furnished by the students.

Wood, preferably pine, is used in the construction instead of metal. Two circular plates are mounted independently upon the same axis, so that the upper plate may revolve upon the lower. The lower plate is 9″ in diameter by ¾″ in thickness, and the upper plate is 7¼″ in diameter by 1″ in thickness. They are pivoted with a small bolt, the head of which is countersunk a quarter of an inch below the surface of the upper plate, while the nut is made flush in the bottom of the lower plate. The large protractor is fastened between the two plates as shown in the drawing. Two protractors may be used but in this case a piece of sheet metal was used to level up the surface on which the top plate revolves, thus eliminating the cost of one protractor. Supports are attached to the upper plate to support the "telescope" which moves upon a horizontal axis. The supports are 5½″ x 2″ x ¾″ and are mortised into the upper plate. The "telescope" contains no lenses, but is simply a piece of brass tubing 8″ by 1¼″ in diameter. Across one end of this tube a piece of sheet metal is soldered and a very small hole is punched in the exact center for an eyepiece. Across the opposite end two fine wires are soldered at right angles to each other (not shown in the drawing). The axis in which this "telescope" is mounted is a piece elliptical in shape, 5½″ long, 2¼″ in greatest diameter, and turned down to ¾″ in diameter at both ends where it is mounted in the supports. A hand indicates the angle of elevation on the small protractor which has been mounted on one of the supports. The pocket compass is inset over the head of the pivot bolt. Across the compass a thread is stretched to indicate the bearing of the instrument. Old clock hands or other pointers are used to indicate the readings on the horizontal limb, one being set at 180 degrees from the one shown in the illustration. The three small spirit levels are mounted as shown. An ordinary amateur camera tripod makes an excellent tripod for this transit. To mount the instrument a piece of metal 3″ diameter and ¼″ in thickness is fastened in the exact center on the bottom of the lower plate. In the center of this a hole is threaded to fit the screw on the tripod. If none such tripod is avail-

able, one can be made by a student in his woodshop course. A coat of black paint gives the instrument a finished appearance.

Equipped with such a transit as has been described, the student can feel that he is already an embryo engineer applying his mathematics to real life situations, and the teacher can know that he is learning much about angles, triangles, parallel lines, circles, and proportion, and that he is coming to realize that mathematics is a vital part of life rather than a series of mental gymnastics prescribed by those who in some mysterious way have charge of the distribution of high school diplomas.

A sound schooling should teach manner of thought rather than matter. It should have a dual aim—to equip a man for hours of work, and for hours of leisure. They interact; if the leisure is misspent, the work will suffer. As regards the first, we cannot expect a school to purvey more than a grip of general principles. Even that is seldom given. The second should enable a man to extract as much happiness as possible out of his spare time. The secret of happiness is curiosity. Now curiosity is not only not aroused, it is repressed. You will say there is not time for everything. But how much time is wasted! Mathematics. . . . A mediæval halo clings round this subject which, as a training for the mind, has no more value than whist-playing. I wonder how many excellent public servants have been lost to England because, however accomplished, they lacked the mathematical twist required to pass the standard in this one subject? As a training in intelligence it is harmful: it teaches a person to underestimate the value of evidence based on their other modes of ratiocination. It is the poorest form of mental exercise—sheer verification; conjecture and observation are ruled out. A study of Chinese grammar would be far more valuable from the point of view of general education. All mathematics above the standard of the office boy should be a special subject, like dynamics or hydrostatics. They are useless to the ordinary man. If you mention the utility of a mathematician like Isaac Newton, don't forget that it was his pre-eminently anti-mathematical gift for drawing conclusions from analogy which made him what he was. And Euclid—that frowsy anachronism! One might as well teach Latin by the system of Donatus. Surely all knowledge is valueless save as a guide to conduct? A guide ought to be up to date and convenient to handle. Euclid is a museum specimen. Half the time wasted over these subjects should be devoted to daughtsmanship and object-lessons. I don't know why we disparage object-lessons; they were recommended by people like Bacon, Amos Comenius and Pestalozzi. They are far superior to mathematics as a means of developing the reasoning powers; they can be made as complex as you please; they discipline the eye and mind, teach a child to discriminate between the accidental and the essential, and demand lucidity of thought and expression.

—From *South Wind* by NORMAN DOUGLAS

Maintenance Drills in the Junior High School

By

DORRIS MAY LEE, *Glendale City Schools, Glendale, California*

and

J. MURRAY LEE, *Director of Research, Burbank, California*

A SURVEY TEST in arithmetic given in a junior high school in Burbank, California, several years ago showed the same condition that many schools have to face—pupils in seventh and eighth grade classes were weak in fundamentals. That such a situation should continue was surely not desired either by the administration or the teachers. The problem, then, was to provide a means of maintaining those computational skills previously learned but usually quickly forgotten. This problem was by no means a local one, as is shown by a statement made by Benz before the Mathematics Section of the Ohio State Educational Conference in 1928:

"We have labored long and faithfully to teach arithmetic properly in the elementary grades and then have promptly proceeded to nullify our efforts by allowing the skills acquired to deteriorate through disuse."[1]

At Burbank, the proposed remedy was that mathematics teachers should give drill in arithmetic fundamentals, each week. An attempt to follow this suggestion was made by the teachers, but many difficulties were encountered. The exercises were prepared by the teachers from day to day, which meant that haphazard drill was given. Such a scheme provided no distribution of drill upon the various combinations or the various types of problems. There was also no continuous plan from grade to grade to insure growth. Much teacher-time was wasted without any guarantee of worth-while results. It was decided after trying this hit-and-miss method that a more thorough study of the problem was needed.

A careful survey (1) of the literature available on drill and (2) of their own needs in the classroom led to the compiling of a list of

[1] H. E. Benz, "The Maintenance of Fundamental Arithmetic Skills in the Junior High School," MATHEMATICS TEACHER, October, 1928, p. 329.

necessary specifications. These specifications have to do with drill material for maintaining acquired skills.[2]

1. Drills should be on the entire process.
2. Drills should come frequently and in small amounts.
3. Each drill unit should be a mixed drill.
4. Drills should have time limits.
5. Drills should have accuracy standards.
6. Drills should provide for a continuous diagnosis of difficulties for individual pupils and for the class as a whole.
7. Drills should provide for retesting, following the remedial teaching which has taken place on the basis of the diagnosis.
8. Drills should provide a measure of the progress made on the retest as a motivating factor for pupils and teacher.
9. Drills should be based on material that has been previously acquired.
10. Drill should be distributed so the emphasis is placed on the relatively more difficult combinations.

Available commercial drill materials were studied to find those that met these specifications. None of the materials met our needs. The main lack in the materials available was in harmonizing items 3 and 6: "Each drill-unit should be a mixed drill," and "Drill should provide for a continuous diagnosis of difficulties for the individual pupils and for the class as a whole." Mixed drill material could be found and diagnostic material could be found, but the two were not combined in an acceptable manner.

The real problem lay in combining mixed drill and a method of diagnosing pupil difficulties. That each is necessary and vital to a modern maintenance drill program has been proved by research and experience.

A study of the value of mixed versus isolated drill made by Repp[3]

[2] Specifications 1-5 are taken from Knight, F. B. "According to What Criteria Should Drill be Organized?" *Third Yearbook,* Department of Superintendence, National Education Association, 1925, pp. 63-91. Knight lists eight criteria of which we accepted these five directly and also his eighth: "Drill should facilitate diagnosis." However, his sixth and seventh, "Examples in a unit of drill should be in order of difficulty" and "Drill units should include verbal problems" were rejected. These were rejected due to the fact that we could not provide for them and also for our specifications number six and seven which we felt were much more essential to a maintenance drill program in the fundamental skills.

[3] Repp, A. C., "Mixed Versus Isolated Drill Organization," *Twenty-ninth Yearbook of the National Society for the Study of Education.* Bloomington, Illinois: Public School Publishing Company, 1930, pp. 535-550.

has shown that mixed drill is definitely superior to "bunched" practice. The diagnostic feature of maintenance drill has been somewhat neglected. However, it needs no involved research to prove that it is wasteful to permit pupils to continue making habitual errors. It seems only common sense that drill should provide a means of detecting these errors. Once they are located, effective remedial teaching may be done. Then a retest should be given to measure the amount of progress achieved. In other words, a good maintenance drill situation should provide for Testing—Diagnosis—Remedial Teaching—Retesting.

A means whereby mixed drill could be made to provide a continuous diagnosis of pupil difficulty was suggested by one of the teachers. This suggestion led to the development of a set of maintenance drill material to meet the above specifications.

To illustrate how such material can be constructed it is necessary to give a brief description.

The drill material[4] consists of twenty drills and twenty tests for each half-year of grades four through eight. Each drill in grades seven and eight contains ten problems and is followed by a test containing problems of the same type and difficulty. The drill is to be given the first of the week and the test at the last. These drills are planned to cover twenty of the most important processes every two weeks, with two problems in each process. Each process has been analyzed into its various parts, and the drill has been so planned as to cover each part. This means that the pupil receives the necessary review with a minimum of problems.

Following are given the twenty different processes on which drill is given. Notice that the number beside each process is the number of that type of problem throughout all the drills and tests.

Odd Numbered Drills and Tests, Nos. 1, 3, 5, etc.

1. Addition of Whole Numbers.
2. Multiplication of Whole Numbers.
3. Addition of Decimals.
4. Multiplication of Decimals.
5. Addition of Fractions.
6. Multiplication of Fractions.
7. Addition and Multiplication of Denominate Numbers.
8. Changing Decimals to Fractions and Fractions to Decimals.
9. Recognizing the Value of Per Cents, Decimals, and Fractions.

[4] *Lee Maintenance Drills and Tests in Arithmetic*. Southern California School Book Depository. Los Angeles, 1930.

10. 7th grade—Changing the Form of a Fraction.
    8th grade—What Per Cent One Number Is of Another.

Even Numbered Drills and Tests, Nos. 2, 4, 6, etc.

11. Subtraction of Whole Numbers.
12. Division of Whole Numbers.
13. Subtraction of Decimals.
14. Division of Decimals.
15. Subtraction of Fractions.
16. Division of Fractions.
17. Subtraction and Division of Denominate Numbers.
18. Changing Per Cents to Fractions and Fractions to Per Cents.
19. Changing Per Cents to Decimals and Decimals to Per Cents.
20. Finding the Per Cent of a Number.

The previous difficulty in mixed drill has been that there was no means for the teacher to diagnose the processes on which the individual had difficulty or to discover on which process it would have been valuable to have given remedial drill to the class as a whole. The distinctive feature of having each process retain the same number throughout all the drills and tests corrects this difficulty. This scheme makes it possible to provide a "Diagnostic Chart" on which the pupil checks the type of problem he misses. Soon a meaningful picture develops on each pupil's chart and it becomes evident to the pupil and also to the teacher the processes on which he needs special help. To facilitate this diagnosis, a teacher's manual is provided, in which is to be found the detailed analysis of each problem in each set of drill and test.

It can readily be seen that this feature makes it possible to have the advantages of diagnosis which drill on one process affords, while keeping the many other advantages of mixed drill.

By use of a class diagnostic chart, the weak spots of the class may be discovered and the required remedial teaching may be directed at these specific weaknesses. The fact that the test contains the same type of problems as the drill then gives a check on the instruction. In actual practice the pupils are eager to correct mistakes they have made on the drill in order to raise their scores on the test. This desire creates a learning situation that is very difficult to obtain in any teaching situation. It provides the opportunity in which remedial teaching will prove most effective.

Through the use of a pupil progress chart: "My Progress Record," an incentive for improvement is furnished the pupil. The pupil is competing against his own record which fact develops and stimulates

him. The number right on each drill was changed to a percentile score; the same percentile score is used for the test. This method equates the drills so that a score of 50 indicates that the pupil did just as well as fifty per cent of the pupils on which the drills were standardized. These records are graphed on the progress chart, using a dotted line for the drill and a straight line for the test. If a pupil maintains his progress evenly his record will be an even line, if he is progressing better than the original group, his line will go upward, if he is not doing so well, it will drop.

The pupil is immediately aware of successful work in the gain of the test score over the drill score that is shown by the chart. It has been shown[5] that the use of the usual progress charts as a means of motivation is advantageous, especially in the case of pupils who are most successful, though very little advantage in case of the slower pupil. This progress chart, by graphing the drill and test scores together, provides immediate success for all pupils. It is not the superior pupil, but rather the pupil who is accustomed to failure, who needs the extra motivation. Through competition with his own drill score each pupil is motivated to raise his test score, thus success meets the efforts of all pupils instead of only a few. The pupil's awareness of success is also continuous, not spasmodic.

Summary

The need for maintenance drill material in arithmetic, as arising from a classroom situation, has been shown. A list of ten specifications that maintenance drills should meet in order to conform to what research and experience have shown to be necessary, have been presented. A means of constructing drill material to meet two of the specifications which have seemed mechanically irreconcilable, namely mixed drill and a continuous diagnosis, has been illustrated. Maintenance drill material has been built which provides for Testing—Diagnosis—Remedial Teaching—Retesting.

[5] I. Panlasigui and F. B. Knight, "The Effect of Awareness of Success and Failure," *Twenty-Ninth Yearbook of the National Society for the Study of Education*. Bloomington, Illinois: Public School Publishing Company, 1930, pp. 611-619.

A Seventh Grade Mathematics Class Bank

By GEORGIA DAY ROBERTSON
Morehead State Teachers College
Morehead, Kentucky

ONE OF THE variations from our regular classroom methods this year, which proved unusually interesting to both pupils and teacher, was our way of handling the unit in banking in the seventh grade of the Training School Junior High. Mathematics teachers, who are looking for schemes to quicken and lighten the steps of boys and girls on their way to the mathematics room, may be interested in our little device and it can be used without any equipment beyond the ordinary classroom.

About a week before we were ready to begin the unit on Banking, of which the work in Interest formed a major part, it was suggested to the group that we create a bank in our class for the purpose of going through the different banking processes as we studied them. A small prize was offered to the pupil who would bring in the most suitable name for this bank. Most of the group handed in names and the prize of an eversharp pencil went to the girl who proposed the name Breckenridge School Bank, Breckenridge being the name of our new Training School building.

A small practice room off the large recitation room was used for the bank, although a corner of the main room would serve the purpose as well. A table set before the door, two chairs behind it for the cashiers, a notebook for depositors' accounts, mimeographed checkbooks, deposit slips, and paper money made up the bank. One of the boys in the class, who stood high in mechanical drawing, stencilled the checks, deposit slips and most originally-designed bills in one-, five- and ten-dollar denominations. All bore the name of the bank. Small change was cut from cardboard and stamped in a design and color hard to counterfeit.

The morning the bank was ready to open, each pupil was given a copy of the following assignment which was intended to be self-explanatory to the pupil, and hence needs no explanation here. When the assignment had been distributed, the teacher spent a short time

in explaining the plan by which the bank and the assigned work were to be connected.

Unit Assignment in Banking—Mathematics 7B
April 2, 1931. Maximum time—two weeks

Read the first page carefully and find out what you are to get from this assignment before *beginning on the exercises. When working on each group, be sure to connect it with the ability which it is to help you develop and write the number of the ability with the number of the group at the top of the first page of your work.*

Depositors in the Breckenridge School Bank may withdraw their money at the close of the banking period only after they have shown that they possess the following abilities.

1. The ability to write a check correctly.
2. The ability to properly endorse a check.
3. The ability to recognize deposit slips, notes, drafts, travelers' checks, certified checks and postal money orders.
4. The ability to open both savings and checking accounts at a bank.
5. The ability to compute interest on money.
6. The ability to find time between dates.
7. The ability to fill out an application for a postal money order.

You will have an opportunity to develop these abilities (1) in the use of the school bank during banking hours, (2) in your trip to the local bank where you may ask questions as well as listen to the banker explain how to use a bank, (3) in working your assigned exercises thoughtfully and carefully, and (4) in asking questions of people who borrow, loan or otherwise handle considerable amounts of money.

(M) Group I. Page 270:* Ex. 1-6. Banking Questions.
(M) II. Page 250: Ex. 1-10. Finding Interest.
III. Page 252: Ex. 1-10. Taking an Interest.
(M) IV. Write a check signed by yourself, the stub filled in, and the check properly endorsed for deposit. Make up your own figures.
V. Page 255: Ex. 1-10. More practice in Interest.
(M) VI. Pages 257-258. Ex. 1-8. Finding the Time Between Dates.
VII. Pages 258-259: Ex. 9-16.
VIII. Write a 200-word article on "Why every boy or girl should have a savings account."
(M) IX. Pages 261-262: Ex. 1-5. Review.
(M) X. Go through the process of borrowing money from, or loaning money to, some member of the class. Use the proper business papers, figure the interest due when the note is paid and complete the transaction to the paying of the note. Write this up with all problems and papers involved.

* Edgerton and Carpenter

XI. Page 264: Ex. 1-10. Word Problems.
(M) XII. Page 265: Ex. 1-15. Review of Interest.
XIII. Write a convincing article of not more than 200 words on when it is good business to borrow money.
(M) XIV. Page 274: Ex. 1-8. Paying by Mail.
(Five problems involving interest brought from home and correctly solved may be substituted for any of the above groups of exercises from the book.)

If the pupils are to do any banking, there must be some way to get the money into their hands and into circulation. We used the following plan as the best scheme we could think of at the time. The group of exercises marked M were to be done first. When a group was finished it was to be handed in with the group number and the corresponding ability number at the top of the first page. For each exercise the pupil had correct both in principle and result, he was to receive one dollar of the paper money. For each one incorrect which he corrected, he was to receive twenty-five cents. After the minimum assignment was completed, he might begin on the other groups for which he could receive two dollars for each correct exercise, and fifty cents for each one corrected. When he received his money he could deposit it in the bank at the very profitable rate of 6 per cent for every 24 hours. (Provided he computed the interest himself and brought the proper result to the cashier each day to be entered on his account.) Any time that he made an error in computing the interest on his balance of the preceding day, he was to lose his income for that 24 hours!

The remainder of the first period was given to the reading of the assignment by the individual pupils and to the beginning of the work. Following our usual custom, no advance explanations or demonstrations of the new work were given. Each pupil was given his opportunity to study out the procedure for himself.

Each day during the two weeks, we had two "checkers" and two cashiers on duty during the class period. The "checkers" marked the papers as they were handed in and paid out the money. For this service they received three dollars each per day. The cashiers received the deposits, entered them in the individual pass books which had been given to each pupil with his first deposit, entered the interest if it had been figured correctly, and balanced the books at the end of each period. For this they received six dollars each per day. The "checkers" were promoted to cashiers one at a time so that we might always have in each position, one who had had one day's experience on the job.

And they needed it, for never were exercises turned in with such promptness and enthusiasm, and a line of impatient depositors clamored continuously for attention at the bank window. At the end of each hour, both "checkers" and cashiers were badly fagged.

Not more than fifteen minutes of any period were given to group instruction, and this only when errors on the papers which had been turned in the day before showed that such instruction was needed. The papers were looked over carefully by the teacher to determine type of errors after the "checkers" were through with them. Group work was based on the common errors and upon any advance directions necessary. The remainder of the sixty-minute period was spent in directed work on the assignment and in the actual processes of banking.

One day was spent in visiting the local bank, where the cashier, by pre-arrangement, showed the pupils the various papers mentioned in their assignment and explained to them certain phases of banking closely connected with their present study.

The basis for marking the pupils on this unit of work was the amount of money each had on deposit at the close of the unit. This plan had been announced at the beginning, and the tendency was what had been expected—hurried, careless work with only the idea of getting a lot of exercises worked in mind. However, after a few disappointing experiences handing in a long list of exercises and receiving only two or three dollars for correct ones, each pupil discovered for himself that only thoughtful, accurate work brought large returns. And it was so much more exciting to have one's hand full of a sheaf of bills even though they were only paper and to watch one's bank account grow, than it was to get a mere red or blue letter at the top of one's paper, so they just had to work their exercises correctly! This, together with the fact that we constantly placed the emphasis on the abilities to be developed, offset the possible danger of working only for the reward and still gave them all the fun of having one.

A check-up at the end of the year—some three months after the unit was completed—showed the group as a whole more familiar with this material than with much that had been learned through the more conventional methods.

William Oughtred

1574-1660

OUGHTRED'S LIFE was long but apparently uneventful. Educated at Eton and at Cambridge, he went into the ministry and for half a century was rector of the country parish of Albury in Surrey. Accordingly his work in mathematics was by way of an avocation, but through his private teaching, his writings and his inventions, he is an important figure in the history of mathematics, especially in its relations to the work of the secondary school.

His biographer, Aubrey, reports that neighboring clerics characterized him as a "pittiful preacher" but that his sermons improved under the threat of losing his living in the troubled times of the Civil War in England. He succeeded in retaining his position, however, and tradition has it that he died of joy at the restoration of Charles II in 1660. In quoting this story, De Morgan says, "It should be added, by way of excuse, that he was eighty-six years old."

In an *Apologeticall Epistle* of 1633, Oughtred describes his quiet life with his infrequent journeys to London about a score of miles away. He says: "Indeed the life and mind of man cannot endure without some interchangeablenesse of recreation, and pawses from the intensive actions of our severall callings; and every man is drawne with his owne delight. My recreations have been diversity of studies; and as oft as I was toyled with the labour of my owne profession, I have allayed that tediousnesse by walking in the pleasant and more then Elysian fields of the diverse and various parts of humane learning, and not the Mathematics onely."*

At Albury, Oughtred gave gratuitous lessons in mathematics to a succession of students among whom were John Wallis, who later held the Savilian professorship of geometry at Oxford, and Sir Christopher Wren, the architect of St. Paul's. Oughtred's theories of education emphasized three points: the appeal to the eye through the liberal use of symbolism, emphasis on rigorous thinking, and the postponing of the use of mathematical instruments and other devices until the logical foundations of the work have been mastered. His writings illustrate

* Florian Cajori, *William Oughtred, A Great Seventeenth-Century Teacher of Mathematics*, Chicago, 1916, p. 9.

these theories. Although mathematical symbolism was little known at that time, he used over one hundred and fifty symbols of which those for multiplication, proportion and the absolute value of a difference (x, ::, ⌣) are the only ones which survive in works in mathematics today. He used the expression $\frac{\pi}{\delta}$ for the approximate ratio of the circumference of a circle to its diameter, and he denoted trigonometric functions by the letters *s, t, se,* for sine, tangent and secant, the other functions being *s co, t co,* and *se co.* He had no knowledge of exponents but wrote the powers of an unknown quantity as *A, Aq, Ac, Aqq, Aqc, Acc.*

His *Clavis Mathematicae* (1631) was a true key to mathematics in its treatment of arithmetic and algebra. In it methods of abridged computation with decimals are shown, and approximate roots of equations of the third, fourth, and fifth degree are found.

The *Circles of Proportion* (1632)** described Oughtred's circular slide rule which seems to have been invented ten years earlier but, there is reason to believe, with no connection to the circular slide rule described by Oughtred's former pupil, Delamain, in 1630. In discussing Delamain's work, Oughtred says: "In reading it . . . I met with such a patchery and confusion of disjoynted stuffe, that I was stricken with a new wonder that any man should be so simple, as to shame himselfe to the world with such a hotch-potch." The invention of the straight slide rule is unquestionably due to Oughtred.

In the *Trigonometrie* (1657), Oughtred divides a circle into 100 equal parts as was suggested in 1585 by Stevin, a scheme used by certain of Oughtred's contemporaries also. Cajori suggests that the wide use of Vlacq's tables, which were on the 360 degree basis, prevented the adoption of this other idea.

Besides these publications, Oughtred maintained an extensive correspondence with mathematicians both in England and abroad and wrote on dialing and gaging, and translated Leurechon's popular work on mathematical recreations.

** Florian Cajori, "Oughtred On the Slide Rule," *Source Book in Mathematics,* edited by D. E. Smith, New York, 1929, pp. 160-164. See also Florian Cajori, "On the History of Gunter's Scale and the Slide Rule during the Seventeenth Century," *University of California Publications in Mathematics,* vol. I, No. 9, pp. 187-209.

NEW BOOKS

Calculus. By Egbert J. Miles of Yale University, and James S. Mikesh of the Lawrenceville school. New York: McGraw-Hill Book Co. Pp. 638 + xiii.

This book is evidently intended by the authors as a first book in the subject. They have modified the symbolism usually employed in text books on calculus, so as to give the student the impression of following his algebraic work in new fields, where a study of rates of change is the center of attention. Thus, the symbol Δx is not encountered, nor is $\frac{dy}{dx}$ explained as the limiting value of $\frac{\Delta y}{\Delta y}$ or $\frac{f(x+\Delta x)-f(x)}{\Delta x}$ but instead, the student encounters an explanation of the meaning of the derivative of $f(x)$ employing

$$f^1(x) = \lim(x \to x_1)\{f(x)-f(x_1)\}/(x-x_1).$$

To one whose training has accustomed him to the conventional symbolism, this departure may seem somewhat troublesome; but to the student taking up the subject for the first time, it need offer no difficulties. The older symbolism is briefly explained on page 63, so that the student can read texts employing the conventional symbols.

The authors have certainly emphasized functionality throughout the text. The student is continually studying problems and applications concerned with the change that has taken place in the function compared with the change that has taken place in the variable. Thus the definite integral

$$\int_a^b f(x)$$

is interpreted:

"If the rate of change of a function $u(x)$ is known to be $f(x)$, then the change $u(b)—u(a)$ produced in $u(x)$ as x changes from $x=a$ to $x=b$ is tween limiting ordinates and the x

$$\int_a^b f(x)."$$

Again, the area under a curve between limiting ordinates and the x axis is regarded as the *change* of area swept over by the generating ordinate as this ordinate changes from $f(x^1)$ to $f(x)$.

This emphasis on functionality, on relative rates of change, certainly gives the student an inkling of what calculus is all about more quickly than most of the older texts on the subject.

Another feature of the book which deserves commendation from the standpoint of sound teaching procedure is the immediate and varied application of every new section of theory to practical problems in geometry, mechanics, physics, and mensuration. The real meaning of the theory thus emerges in the mind of the student. He masters it through *use*, and not as an abstract system of symbols, whose slight meaning content quickly evaporates.

From the same standpoint, the book

deserves approval, because the formulae of differentiation and integration are introduced very gradually, and numerous applications are given after only a few fundamental forms have been presented. Integration by parts is presented and applied to algebraic functions before a single trigonometric function has been differentiated or integrated. By the time the student reaches some of the complex forms and special devices, the general meaning, and the skill in practice that have been built up make this work easy.

Many phases of theory are introduced where they are needed to solve interesting problems which call for their use. Thus, McClaurin's series is developed as part of the study of approximations.

In general, the book emphasizes meaning and minimizes mere forms and manipulations. It is a mine of material for any one who is evolving a first course in calculus for the last year of the high school, although it is too extensive and difficult for use as a text there. It is an excellent college text, and should appeal to those who value meaning more than form and convention.

W. S. SCHLAUCH
New York University

The Rhind Mathematical Papyrus. By Chancelor A. B. Chace, Brown University.

A high school student of mathematics, whenever he begins to think things through for himself, must at some time wonder how the human race learned to count and how arithmetic and algebra came to fill so important a place in the affairs of men. There are few chapters in ancient or modern history dealing with peaceful methods of upbuilding civilization that are as pathetic and even tragic as the struggle of mankind to build a number system adequate for the solution of the world's problems.

The Rhind Mathematical Papyrus is a document written by an Egyptian priest between three and four thousand years ago, which shows one stage (the oldest on record) of that struggle of men to think in terms of numbers. This Papyrus is preserved in the British Museum where very few, if any, of our high school boys and girls would ever be likely to see it, and even if they did it would not be possible for them to decipher the writing which is in Egyptian hieratic symbols. But fortunately it has been photographed and translated so that any one interested can see just how it looks in its original form and can compare the hieroglyphic writing with both the literal and the free translations in English. The Papyrus is about 18 feet long and 13 inches wide and is reproduced in 31 photographic plates together with 109 facsimile plates each showing a small portion magnified so as to bring out clearly both the original handwriting of the priest who transcribed it and the corresponding transliteration in the more common hieroglyphic form. On the pages opposite these 109 plates are explanatory notes and a complete literal translation in English. All these plates are bound in a massive volume ten by fourteen inches in size. A second volume contains a wealth of descriptive matter, an extensive bibliography of Egyptian mathematics, and a free translation of the whole Papyrus.

A study of this document by high school students of mathematics will not only be a revelation of the difficulties encountered by mankind in developing arithmetic and algebra, but it will also furnish a most interesting exercise for

members of a school mathematics club to work out the Egyptian method of solution of some of the problems and compare this with our present day algebraic methods. This will develop a wholesome appreciation of what has been accomplished in mathematics since this earliest known textbook was compiled some 3,600 years ago.

It is my belief that it is well worth while for all students of algebra (and arithmetic) to have access to these volumes, but it may require a little careful planning to make this possible, since the present price of the set is $20.00 (an amount far below the actual cost of preparation and printing). However, I believe it can be done in one of the three following ways:

(1) In a city having two or more high schools, let the mathematics teachers petition the board of education to purchase the set, keep it at headquarters, and loan it out for a specific time to each school during each year. In case there is only one high school in the system it then could be kept in the school library. If there are several high schools, more than one set may be needed.

(2) If this petition fails, try to make an arrangement with the public library so that pupils could have access to these volumes in some private room or reference room.

(3) If both these plans fail, then the mathematics club can surely devise means whereby the money can be raised. It may be by some entertainment or by direct contributions to a book fund or a class might present the set to the school library as a memorial. There are numerous mathematical plays which have been published from time to time. One of these might be elaborated and presented to the whole school with a small admission fee. Something can be done if the problem is tackled in earnest.

There is one admonition, however, which should be faced. The edition of the Papyrus sets was limited to 600 and a large number of these have been sold. When this edition is exhausted no more can be had. Furthermore, as they become scarce the demand will increase and the price will be advanced —as is usual in such cases. In fact the price will be raised to $25.00 on January 1, 1932. Hence action should be taken at once by those who wish to take advantage of the present price of $20.00.

The Papyrus was published under the auspices of the Mathematical Association of America but is distributed to non-members of this Association by the Open Court Publishing Company, 337 East Chicago Avenue, Chicago, Illinois.

H. E. Slaught
University of Chicago

Sagan om de 10 technen. By K.-G. Hagstroem. Stockholm, A. Bonniers Förlag, 1931. Pp. 147. Price 5.75 kronor (approximately $1.56 in Sweden).

This recent work by Dr. Hagstroem will be of special interest to Swedish readers, of whom we have many in the middle west. It is, however, of great value to all who are concerned with the history of mathematics, even if relatively few are familiar with the language in which it is written. This value lies in the numerous illustrations, many of which have never before appeared in any history of the subject and which tell in a pictorial way the story of our ten numeral symbols. Indeed, it is quite of the nature of a visual source-book of the development of these bases of arithmetic.

Dr. Hagstroem approaches the subject with a linguistic equipment which is becoming very rare in our country, and with a due appreciation of the meaning of scholarly research. The book is one which may well find a place in the reading rooms of our secondary schools as well as in public and university libraries.

DAVID EUGENE SMITH

Publications Received

Accounting, Principles and Procedure. Groggin, Walter J. and Toner, James V. Houghton Mifflin Company, 2 Park Street, Boston, Mass. $1.80.

Agricultural Mathematics. Plant, L. C. McGraw-Hill Book Company, 199 pp. $2.50.

Algebra for Junior and Senior High Schools. Calhoun, J. W., White, E. V., and Simpson, Jr., T. Mc. N. Johnson Publishing Company, New York, N.Y.

Applied Business Arithmetics. Steele, Charles E., and Muench, George W World Book Company, Yonkers-on-Hudson, N.Y.

A Second Course in Algebra. Barber, H. C. Houghton Mifflin Company, 2 Park Street, Boston, Mass. $1.40. 505 pp.

Exercises and Tests in Algebra through Quadratics. Smith, David Eugene, Reeve, William David, and Morss, Edward Longworth. Ginn and Company, 15 Ashburton Place, Boston, Mass. $.40.

Geometry Workbook. Kingsbury, H. B., and Wallace, R. R. Bruce Publishing Company, Milwaukee, Wis.

Higher Arithmetic. Smith, Frederick A. 608 Dearborn Street, Chicago, Ill.

Laboratory Manual to Accompany Accounting. Groggin, Walter J., and Toner, James V. Houghton Mifflin Company, 2 Park Street, Boston, Mass. $1.80.

Lustiges und Merkwudiges Von Zahlen und Formen. Dr. Walther Lietzmann. Ferdinand Hirst, Breslau.

The New Day Arithmetic, Elementary Book. Durell, Fletcher, Gillett, Harry O., and Durell, Thomas J. Charles E. Merrill Company, New York, N.Y.

The New Day Arithmetic, Intermediate Book. Durell, Fletcher, Gillett, Harry O., and Durell, Thomas J. Charles E. Merrill Company, New York, N.Y.

New Rapid Exercises in Rapid Calculation. Powers, Earle, and Loker, Harold W. Ginn and Company, 15 Ashburton Place, Boston, Mass. $.96

Practical Mathematics, Part I—Arithmetic. Palmer, Claude Irwin. McGraw-Hill Book Company, Inc., 370 Seventh Avenue, New York, N.Y. $1.25.

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Raleigh, Clark, John R., and Lindell, Selma A. World Book Company, Yonkers-on-Hudson, N.Y. $1.36.

Solid Geometry. Seymour F. American Book Company, 88 Lexington Avenue, New York, N.Y.

Solid Geometry. Haertter, Leonard D. The Century Company, New York, N.Y.

Source Book in Mathematics. Smith, David Eugene. McGraw-Hill Book Company, 370 Seventh Avenue, New York, N.Y. 701 pp. $5.00.

Statistical Tables and Graphs. Mudgett, Bruce D. Houghton Mifflin Company, 2 Park Street, Boston, Mass. 194 pp. $1.75.

Strayer-Upton Junior Mathematics, Modern Algebra, Ninth Year. Upton, Clifford Brewster. American Book Company, 88 Lexington Avenue, New York, N.Y. $1.20.

Survey of College Entrance Credits and College Courses in Music. Prepared by the Research Council of the Music Supervisors National Conference in Cooperation with the National Bureau for the Administration of Music. National Bureau for the Advancement of Music, 45 West 45th Street, New York, N.Y.

Washburn Individual Arithmetic. Books One to Twelve. Washburn, Carlton, Koepke, Emma Jaycox, McAfee, Clauda Rogers, and Barnett, Frieda. World Book Company, Yonkers, N.Y.

Washburn Individual Arithmetic. Correction Books, One to Five and Six to Twelve. Washburn, Carlton, Koepke, Emma Jaycox, McAfee, Clauda Rogers, and Barnett Frieda. World Book Company, Yonkers, N.Y.

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Workbook on Plane Geometry. McIntyre, Hugh D. World Book Company, Yonkers, N.Y.

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